PARTIAL FRACTIONS

The goal of the section is to integrate rational functions (a ratio of polynomials), by expressing it as a sum of simpler fractions, called partial fractions, that we already know how to integrate.

Example.

$$\frac{3x-1}{x^2-2x-3} = \frac{1}{x+1} + \frac{2}{x-3}$$

Then

$$\int \frac{3x-1}{x^2-2x-3}\,dx = \int \frac{1}{x+1} + \frac{2}{x-3}\,dx$$

$$= \ln|x+1| + 2\ln|x-3| + C$$

This example illustrates that if we know how the rational function splits as a sum of simpler functions, we can compute its integral fairly easily. This method is called partial fraction method, and we want do see how the method of partial fractions work in general.

Let

$$f(x) = \frac{P(x)}{Q(x)}$$

be a rational function, that is, $P(x)$ and $Q(x)$ are polynomials.

Definition 1. $f(x)$ is called proper if the degree of $P(x)$ is less than the degree of $Q(x)$.

If $f(x)$ is a proper, then it is possible to express f as a sum of simpler fractions. If $f(x)$ is not proper (called improper), then we can use long division to to rewrite

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ is the quotient of the division and $R(x)$ is the remainder.

Exercise 1. Find

$$\int \frac{x^3+x}{x-1}\,dx$$

Since the rational fraction is improper, use long division to rewrite the fraction as a sum of a polynomial and a proper fraction. The basic integral identity we will need regularly for this section is

Example.

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C}\;; \qquad a \neq 0$$

We can prove this identity by using the substitution $u = ax+b$.

For example,

$$\int \frac{3}{4x-1}\,dx = \frac{3}{4}\ln|4x-1| + C$$

Now given a proper fraction $\frac{R(x)}{Q(x)}$, we want to express it as a sum of simple fractions. The first step in this process is to factor $Q(x)$, and how it splits up depends on the factors of Q. There are 4 seperate cases:

Case 1 WHEN $Q(x)$ IS A PRODUCT OF DISTINCT LINEAR FACTORS

If,

$$f(x) = \frac{x-1}{x(x+1)(2x-1)}$$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x-1}$$

where A, B and C are constants which can be determined as explained in class.

Exercise 2. Find

$$\int \frac{x-4}{x^3-5x^2+6x}\,dx$$

Case 2 WHEN $Q(x)$ IS A PRODUCT OF LINEAR FACTORS, SOME OF WHICH ARE REPREATED.

$$
\begin{aligned}
f(x) &= \frac{x^2 - x + 1}{x^2(x-1)^3} \\
&= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}
\end{aligned}
$$

A, B, C, D and E are constants that need to be determined. The factor $x - 1$ occurs three times in the denominator. We accont for that by forming fractions containing increasing powers of this factor in the denominator. We do the same thing for x, which occurs twice.

Exercise 3. Evaluate

$$\int \frac{3x^2 - 4x + 2}{x^3 - 2x^2 + x}\, dx$$

Before we discuss the next two cases, we need the following definition,

Definition 2. A degree 2 polynomial $f(x) = ax^2 + bx + c$ is called <u>irreducible</u> if $b^2 - 4ac < 0$.

Another integral that we need to be familiar with is of the following type:

Exercise 4. Show that

$$\int \frac{x-2}{x^2+9}\,dx = \frac{1}{2}\ln|x^2+9| - \frac{2}{3}\arctan\left(\frac{x}{3}\right)$$

To solve this integral, we can rewrite it as

$$\int \frac{x-2}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx - \int \frac{2}{x^2+9}\,dx$$

Then first integral can be solved using substitution $u = x^2 + 9$. The second integral can be solved by remembering that

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}\,; \qquad\qquad a \neq 0$$

Case 3: $Q(x)$ CONTAINS IRREDUCIBLE QUADRATIC FACTOR, NONE OF WHICH ARE REPEATED

$$f(x) = \frac{x}{(x-2)(x^2+4)(3x^2+1)}$$

$$= \frac{A}{(x-2)} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{3x^2+1}$$

A, B, C, D and E are constants that need to be determined.

Exercise 5. Evaluate

$$\int \frac{2x^3 - x^2 + 5x + 2}{x^4 + 5x^2 + 4}\,dx$$

Case 4: $Q(x)$ CONTAINS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

$$f(x) = \frac{x^2}{(x-2)(x^2+4)^2}$$

$$= \frac{A}{(x-2)} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

A, B, C, D and E are constants that need to be determined.

The factor (x^2+4) occurs twice in the denominator. We account for that by forming fractions containing increasing powers of this factor in the denominator.

Exercise 6. Evaluate

$$\int \frac{x^4+1}{x(x^2+1)^2}\, dx$$

Exercise 7.

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\, dx$$

Exercise 8. Find

$$\int \frac{(x^4-x^3+11x^2-11x+10)}{(x-1)(x^2+9)}$$

Note that the rational function is not proper.

1) $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

This fraction is not proper so use long division

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x-1 & x^3 + 0x^2 + x + 0 \\
 & \underline{\overset{-}{x^3} \overset{+}{-} x^2} \\
 & \quad x^2 + x + 0 \\
 & \quad \underline{\overset{-}{x^2} \overset{+}{-} x} \\
 & \qquad 2x + 0 \\
 & \qquad \underline{+\,2x - 2} \\
 & \qquad\qquad 2
\end{array}
$$

$$\text{Then, } \frac{x^3+x}{x-1} = \underbrace{x^2+x+2}_{\text{quotient}} + \frac{\overbrace{2}^{\text{remainder}}}{x-1}$$

$$\int \frac{x^3+x}{x-1}\,dx = \int x^2+x+1 + \frac{2}{x-1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + 2\ln|x-1| + C$$

2) Find

$$\int \frac{x-4}{x^3-5x^2+6x}\,dx$$

Factor denominator

$$x^3-5x^2+6x = x(x^2-5x+6) = x(x-3)(x-2)$$

Then, $$\frac{x-4}{x(x-3)(x-2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-2}$$

Multiply both sides by $x(x-3)(x-2)$

$$x-4 = A(x-3)(x-2) + Bx(x-2) + Cx(x-3)$$

<u>Let $x=0$</u>

$-4 = 6A$

$A = -\frac{2}{3}$

<u>Let $x=2$</u>

$-2 = C(2)(2-3)$

$\Rightarrow C = 1$

<u>let $x=3$</u>

$-1 = B(3)(3-2)$

$\Rightarrow B = -\frac{1}{3}$

$$\int \frac{x-4}{x^3-5x^2+6x}\,dx = \int \frac{-2/3}{x} + \frac{-1/3}{x-3} + \frac{1}{x-2}\,dx$$

$$= -\frac{2}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{1}{x-3}\,dx + \int \frac{1}{x-2}\,dx$$

$$= -\frac{2}{3}\ln|x| - \frac{1}{3}\ln|x-3| + \ln|x-2| + C$$

3) $$\int \frac{3x^2-4x+2}{x^3-2x^2+x}\,dx$$

$$x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$$

Then,

$$\frac{3x^2-4x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$ $\rightarrow$ Multiply both sides by denominator

$$3x^2-4x+2 = A(x-1)^2 + Bx(x-1) + Cx$$

$\underline{x=0}$ $\qquad$ $\underline{x=1}$

$2 = A$ $\qquad$ $3-4+2 = C \Rightarrow C = 1$

$\underline{x = 2}$

$$12 - 8 + 2 = A + 2B + 2C$$

$$6 = 2 + 2B + 2$$

$$2B = 2$$

$$B = 1$$

Then,

$$\int \frac{3x^2 - 4x + 2}{x^3 - 2x^2 + x} \, dx = \int \frac{2}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2} \, dx$$

$$= \int \frac{2}{x} \, dx + \int \frac{1}{x-1} \, dx + \int \frac{1}{(x-1)^2} \, dx \longrightarrow \begin{aligned} u &= x - 1 \\ du &= dx \end{aligned}$$

$$\int u^{-2} du = -\frac{1}{u} + C$$

$$= 2 \ln|x| + \ln|x - 1| - \frac{1}{(x-1)} + C$$

4)

$$\int \frac{x-2}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx - \int \frac{2}{x^2+9}\,dx$$

$$= \frac{1}{2}\ln|x^2+9| - \frac{2}{3}\arctan\left(\frac{x}{3}\right) + C$$

Follow instruction on handout.

5) $$\int \frac{2x^3 - x^2 + 5x + 2}{x^4 + 5x^2 + 4}\,dx$$

Factor denominator

$x^4 + 5x^2 + 4$

Jump in power is 2

Set $u = x^2$

So $u^2 + 5u + 4$

$= (u+4)(u+1)$

$= (x^2+4)(x^2+1)$

Both are irreducible quadratic

$$\frac{2x^3 - x^2 + 5x + 2}{(x^2+4)(x^2+1)} = \frac{Ax + B}{(x^2+4)} + \frac{Cx + D}{(x^2+1)}$$

Multiply both sides by denominator

$$2x^3 - x^2 + 5x + 2 = (Ax+B)(x^2+1) + (Cx+D)(x^2+4)$$

$$2x^3 - x^2 + 5x + 2 = Ax^3 + Bx^2 + Ax + B + Cx^3 + Dx^2 + 4Cx + 4D$$

$$2x^3 - x^2 + 5x + 2 = (A+C)x^3 + (B+D)x^2 + (A+4C)x + (B+4D)$$

By comparing coefficients

$A + C = 2$; $B + D = -1$

$A + 4C = 5$; $B + 4D = 2$

Then, solving for A, B, C, D we get

$A = 1$ $\qquad$ $B = -2$

$C = 1$ $\qquad$ $D = 1$

Then,

$$\int \frac{2x^3-x^2+5x+2}{x^4+5x^2+4}\,dx = \int \frac{x-2}{x^2+4}\,dx + \int \frac{x+1}{x^2+1}\,dx$$

$$= \int \frac{x}{x^2+4}\,dx - 2\int \frac{1}{x^2+4}\,dx + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x^2+4| - 2\cdot\frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln|x^2+1| + \arctan x + C$$

$$= \frac{1}{2}\ln|x^2+4| - \arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln|x^2+1| + \arctan x + C$$

6) $$\int \frac{x^4+1}{x(x^2+1)^2}\,dx$$

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply both sides by denominator

$$x^4+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$x^4+1 = A(x^4+2x^2+1) + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex$$

$$x^4+1 = x^4(A+B) + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

Comparing coefficients

$$A+B = 1 \xRightarrow{A=1} 1+B = 1 \Rightarrow B = 0$$

$$C = 0$$

$$2A+B+D = 0 \xRightarrow{A=1,\, B=0} 2+D = 0 \Rightarrow D = -2$$

$$C+E = 0 \xRightarrow{C=0} 0+E = 0 \Rightarrow E = 0$$

$$A = 1$$

Then,

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx = \int \frac{1}{x} + \frac{-2x}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{2x}{(x^2+1)^2}\,dx \longrightarrow \begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}$$

$$= \ln|x| - \int \frac{du}{u^2}$$

$$= \ln|x| - \frac{u^{-1}}{-1} + C = \ln|x| + \frac{1}{x^2+1} + C$$

7) $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

Multiply both sides by denominator

$$x^2-2x-1 = A(x^2+1)(x-1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$x^2-2x-1 = Ax^3 + Ax - Ax^2 - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

$$x^2-2x-1 = x^3(A+C) + x^2(-A-2C+D+B) + x(A+C-2D) + (-A+B+D)$$

Comparing coefficients

$$A+C = 0 \xRightarrow{C=-1} A = 1$$

$$-A-2C+D+B = 1 \xRightarrow{-A+B=-2} -2-2C+D = 1 \xRightarrow{D=1} -2 = 2C \Rightarrow C = -1$$

$$A+C-2D = -2 \xRightarrow{A+C=0} -2D = -2 \Rightarrow D = 1$$

$$-A+B+D = -1 \xRightarrow{D=1} \underline{-A+B = -2} \xRightarrow{A=1} B = -1$$

Then,

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{-x+1}{x^2+1}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| + \arctan x + C$$

8) $$\int \frac{(x^4-x^3+11x^2-11x+10)}{(x-1)(x^2+9)}\,dx$$

Note that the fraction is improper, so we need to do long division. $(x-1)(x^2+9) = x^3-x^2+9x-9$

$$\begin{array}{r|l} & \quad x \\ \hline x^3-x^2+9x-9 & x^4-x^3+11x^2-11x+10 \\ & x^4-x^3+9x^2-9x \\ & - \quad + \quad - \quad + \\ \hline & \qquad 2x^2-2x+10 \end{array}$$

Therefore,

$$\frac{x^4 - x^3 + 11x^2 - 11x + 10}{(x-1)(x^2+9)} = x + \underbrace{\frac{2x^2 - 2x + 10}{(x-1)(x^2+9)}}$$

Need to turn this
into partial fractions.

$$\frac{2x^2 - 2x + 10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+9}$$

Multiply both sides by denominator

$$2x^2 - 2x + 10 = A(x^2+9) + (Bx + C)(x-1)$$

$$2x^2 - 2x + 10 = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$

$$2x^2 - 2x + 10 = (A+B)x^2 + x(-B+C) + (9A - C)$$

Comparing coefficients

$A + B = 2 \rightarrow$ (1)

$-B + C = -2 \rightarrow$ (2)

$9A - C = 10 \rightarrow$ (3)

Add (2) and (3)

$$\begin{array}{rl} 9A - B &= 8 \\ + \quad A + B &= 2 \\ \hline 10A &= 10 \Rightarrow A = 1 \end{array}$$

Then, $A+B=2 \Rightarrow 1+B=2 \Rightarrow B=1$

$-B+C=-2 \Rightarrow -1+C=-2 \Rightarrow C=-1$

Finally,

$$\int \frac{x^4-x^3+11x^2-11x+10}{(x-1)(x^2+9)}\,dx = \int x + \frac{1}{x-1} + \frac{x-1}{x^2+9}\,dx$$

$$= \frac{x^2}{2} + \ln|x-1| + \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$$

$$= \frac{x^2}{2} + \ln|x-1| + \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C .$$